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THE INVERSE PROBLEM OF THE SCATTER OF UNGUIDED MISSILES†

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The inverse scatter problem determines the characteristics of the perturbations of the initial conditions, design parameters, and the wind velocity from given characteristics of the scatter of the point of impact of an unguided missile. A deterministic and a probabilistic formulation of the problem are considered. Some properties of its solutions are noted and a method of computing them is proposed.

ONE OF the main problems in the exterior ballistics of multiple launches of unguided missiles is calculating the scatter of the point of impact of the missile from the known characteristics of the perturbations of the initial conditions, design parameters and the wind velocity. Analytical methods have been developed for solving this problem [1]; computer-based linearization and Monte Carlo methods are also available [2]. This paper examines the inverse of the problem previously considered, namely, the problem of determining the maximum level of perturbations in the initial conditions, design parameters and the wind velocity that ensures a given scatter of the points of impact for any launching conditions.

1. PERTURBATIONS OF THE DESIGN PARAMETERS AND WIND VELOCITY

We will assume that the scatter of the points of impact of the missile is governed by perturbations of the design parameters and by perturbations of the wind velocity and initial conditions at the instant of lift-off. After separation from the launching guide, the missile is treated as an undeformable rigid body that carries a solid fuel charge and moves in the uniform gravitational field acted upon by a constant horizontal wind, the jet propulsion force, the reactive Coriolis and aerodynamic forces, and the corresponding momenta. The perturbed motion of the perturbed missile along the launching guide is described using the simplest model of the missile as a material point of variable mass that moves rectilinearly under the action of a propulsive force.

The unperturbed missile has both mass and aerodynamic symmetry about the longitudinal axis; the propulsive force also acts along this axis. The variables in the unperturbed motion of the unperturbed missile will be denoted by a prime. Each unperturbed motion is uniquely defined by the vector of unperturbed launching conditions $a' = (h', \theta_0', u', \psi_u')$, where h' is the launcher altitude above sea level, θ_0' is the launching angle, u' is the magnitude of the wind velocity $\mathbf{u}'$, and ψ_u' is the angle between $\mathbf{u}'$ and the Ox axis (see below).

In the perturbed missile, the distribution of the mass and the propulsive force is assumed to be slightly asymmetric; aerodynamic asymmetry is ignored. We also allow for small perturbations of all design parameters occurring in the equations of motion of the axisymmetric missile. Let t_* be the time when the perturbed missile starts moving along the launch guide, t_0 the time when the missile separates from the guide, and τ the end of the section under power. The system of coordinates $C_\tau^0 x_2 y_2 z_2$ is rigidly fixed to the missile shell; the origin of this system is the projection of the point C_τ (the centre of mass of the missile under power) on the aerodynamic axis of symmetry, with the $C_\tau^0 x_2$

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axis pointing along the aerodynamic axis of symmetry towards the nose. The propulsive force $\mathbf{P}(t)$ is characterized by its magnitude $P(t)$ and the unit vector $\nu(t)=\mathbf{P}(t)/P(t)$. The moment of the propelling force about the centre of mass is defined by specifying the point of intersection $K(t)$ of the line of action of the force with the bottom section plane. Using the notion of effective length of the launching guide, the actual dependences $P'(t)$ and $P(t)$ are approximated by rectangular ones, which are characterized by three parameters: the mass-flow rate after separation, the effective discharge rate and the propellant mass. The times t'_* and t_* thus deviate from the actual times. Since the mass-flow rate is assumed to be constant, the missile mass is a linear function of t in the time interval $[t_*, \tau]$.

All other design parameters that explicitly depend on t in this time interval (the coordinates of the centre of mass, the components of the central inertia tensor, the coordinates of the point K, and the components of the vector ν in the system $C_\tau^0x_2y_2z_2$) are approximated by functions linear in t and are therefore completely characterized by their values at the instants of time t_* and τ. A change of t_* keeping all other conditions constant does not alter the impact point, and we accordingly take $t_*=t'_*=0$. The perturbations of the exterior missile geometry are allowed for by perturbing the aerodynamic coefficients. They are characterized by small constant values: perturbations of the corresponding "shape factors".

Wind velocity perturbations are considered only on the propulsion section of the flight trajectory where their effect on the impact point scatter is the most significant.

2. THE PERTURBED MOTION OF A PERTURBED MISSILE

Defining in the usual way the variables x, y, z, v, θ, ψ, p, q, r, α, β, γ that describe the missile dynamics, we will combine them into a phase vector W of the system of differential equations of the perturbed motion of a perturbed missile after separation from the launching guide. Here x, y, z are the coordinates of the centre of mass of the missile in the starting system of coordinates $Oxyz$; the Ox axis points horizontally in the firing direction, the Oy axis points vertically upwards; v is the magnitude of the velocity vector $\mathbf{v}$ of the centre of mass, θ is the angle between the Ox axis and the projection of $\mathbf{v}$ onto the Oxy plane, and ψ is the angle between the vector $\mathbf{v}$ and the Oxy plane. The angles α and β are defined in the same way as θ and ψ and describe the direction of the longitudinal axis $C_\tau^0x_2$ in the semi-velocity system of coordinates $C_\tau^0x_0y_0z_0$. The semi-attached system of coordinates $C_\tau^0x_1y_1z_1$ is transformed into $C_\tau^0x_2y_2z_2$ by rotating it through the angle γ about the axis $C_\tau^0x_1$. We shall denote by p, q, r the projections of the absolute angular velocity vector of the missile on the semi-attached axes. We shall assume that the initial perturbations $\Delta\theta_0$, ψ_0, q_0, r_0, α_0, β_0 at the time of separation from the launching guide t_0 and the perturbations of the design parameters and the wind velocity introduced above are independent and small. If the rotational velocity p and the restoring moment coefficient are sufficiently large during the entire flight and the wind velocity is sufficiently small, then if there are no resonance effects ψ, q, r, α and β will be small; the perturbations of these variables and the other components of the phase vector W (with the exception of $\Delta W_{12}=\Delta\gamma$) will also be small in this case.

Thus, ignoring terms of second and higher order smallness in the equations of motion, we obtain a linear (in $\Delta W_1, \ldots, \Delta W_{11}$) non-homogeneous system of differential equations of the perturbed motion of the perturbed missile that defines $\Delta W=W-W'$ for $t\geqslant t_0$. The coefficients of this system depend on the unperturbed motion of the unperturbed missile.

Some of the design parameter perturbations do not occur in this system, i.e. they are unimportant within the framework of the linearized theory. The wind velocity perturbations, the initial perturbations, and the remaining design parameter perturbations form the vector

$$\begin{aligned}\epsilon &= (\epsilon_1^{(1)}, \epsilon_1^{(2)}, \ldots, \epsilon_{11}^{(1)}, \epsilon_{11}^{(2)}, \epsilon_{12}, \ldots, \epsilon_{18}) = \\ &= (\Delta u_x, \Delta u_z, \nu_{y*}, \nu_{z*}, \kappa_{y*}, \kappa_{z*}, y_{K*}, z_{K*}, y_{C*}, z_{C*}, \\ &\nu_{y\tau}, \nu_{z\tau}, \kappa_{y\tau}, \kappa_{z\tau}, y_{K\tau}, z_{K\tau}, y_{C\tau}, z_{C\tau}, q_0, r_0, \\ &\alpha_0, \beta_0, \Delta\theta_0, \psi_0, \Delta\mu, \Delta u_e, \Delta m_1, \Delta m_2, \epsilon_x)\end{aligned} \tag{2.1}$$

Here Δu_x and Δu_z are the projections of the wind velocity perturbation vector onto the Ox and Oy axes. We will denote by ν and κ with the subscripts y and z the projections on the $C_\tau^0 y_2$, $C_\tau^0 z_2$ axes of the vector ν and the unit vector κ pointing in the direction of the principal central axis of inertia with the least moment of inertia. We will denote by y and z with the subscripts K and C the corresponding coordinates of the point K and the centre of mass in the system $C_\tau^0 x_2 y_2 z_2$. The subscripts *, 0 and τ denote the values of the variables at instants of time t_*, t_0, and τ. $\Delta\mu$ and Δu_e are the perturbations of the mass-flow rate and the effective discharge rate, Δm_1 and Δm_2 are the perturbations of the propellant mass and the missile mass on the burnout section of the flight trajectory, and ϵ_x is the perturbation of the shape factor. The system also explicitly contains $\Delta W_7 = \Delta p$. The linearized system of the equations of perturbed motion of a perturbed missile thus has the form

$$
\begin{aligned}
(\Delta w)^{\cdot} &= B(t, a')\Delta w + b_1^{(1)}(t, a')\epsilon_1^{(1)} + b_1^{(2)}(t, a')\epsilon_1^{(2)} + \\
&+ \sum_{j=2}^{9}\sum_{s=1}^{2}[b_j^{(1,s)}(t, a')\cos\gamma + b_j^{(2,s)}(t, a')\sin\gamma]\epsilon_j^{(s)} + \\
&+ b_P(t, a')\Delta P(t, a', \epsilon) + b_m(t, a')\Delta m(t, a', \epsilon) + b_{18}(t, a')\epsilon_{18} \\
\Delta w &= \mathrm{col}(\Delta x, \Delta y, \Delta z, \Delta v, \Delta\theta, \Delta\psi, \Delta q, \Delta r, \Delta\alpha, \Delta\beta)
\end{aligned}
\tag{2.2}
$$

The matrix $B(t, a')$ is continuous, the column vectors $b(t, a')$ have discontinuities at $t = \tau'$, the functions $\Delta P(t, a', \epsilon)$, $\Delta m(t, a', \epsilon)$ are piecewise-continuous in t and are expressed in terms of $\epsilon_{14}, \ldots, \epsilon_{17}$, i.e. in terms of $\Delta\mu$, Δu_e, Δm_1, Δm_2, and the dependence of γ on t is defined by the equation $\gamma^{\cdot} = p$. At the instant of separation $t_0 = t'_0 + \Delta t_0$, we have $\Delta w(t_0) = (0, 0, 0, \Delta v_0, \Delta\theta_0, \psi_0, q_0, r_0, \alpha_0, \beta_0)$ are arbitrarily small perturbations and Δv_0 and Δt_0 are determined from the equations of motion along the launching guide in the form of linear homogeneous functions of the perturbations $\epsilon_{14}, \ldots, \epsilon_{17}$.

3. PERTURBATIONS OF THE IMPACT POINT COORDINATES

If the dependence of $\gamma(t)$ is known, the solution $\Delta w(t)$ of system (2.2) is expressed by a well-known formula in terms of the fundamental matrix $\Phi(t)$ of the corresponding homogeneous system, the initial values, and the non-homogeneous terms. We thus obtain an expression for Δw at the impact time T. This expression includes integrals over $[t_0, T]$ of products of functions of the form $\Phi(T)\Phi^{-1}(t)b(t)$ by $\cos\gamma(t)$ and $\sin\gamma(t)$. The components of these vector-valued functions are similar to the function $R[s_a, \sigma(t)]$ in [1, Sec. 53]. They can take large values and vary rapidly only on the critical section of the flight trajectory immediately after lift-off, while on the rest of the trajectory these functions take relatively small values and vary slowly. Thus, if the angular velocity of rotation p is sufficiently large, it is the critical section that makes the main contribution to these integrals, because on the rest of the trajectory the integrands describe nearly harmonic high-frequency oscillations of small amplitude with a slowly varying frequency and amplitude. The integrals of these functions are therefore negligibly small.

From the equation $\gamma^{\cdot} = p' + \Delta p$ we see that on the critical section, because of its short length and the condition $\gamma(t_0) = \gamma'(t'_0) = 0$ (this condition is ensured by the choice of the $C_\tau^0 y_2$ axis in the system of $C_\tau^0 x_2 y_2 z_2$), we have $\gamma(t) = \gamma'(t - \Delta t)$with a small error and the relevant integrals can be taken approximately. Replacing t_0 and T by t'_0 and T' in the expression for $\Delta w(T)$, we obtain in the linear approximation the following expression for the perturbations $\Delta w_k(T)$ of the coordinates $w_1 = x$, $w_2 = y$, $w_3 = z$ at the impact time $T = T' + \Delta T$

$$
\Delta w_k(T) = \sum_{j=1}^{11}(M_{kj}^{(1)}\epsilon_j^{(1)} + M_{kj}^{(2)}\epsilon_j^{(2)}) + \sum_{j=12}^{18} M_{kj}\epsilon_j \ (k = 1, 2, 3)
\tag{3.1}
$$

Here $\epsilon_j^{(1)}$, $\epsilon_j^{(2)}$ $(j = 1, \ldots, 11)$ and ϵ_j $(j = 12, \ldots, 18)$ are arbitrarily small perturbations (2.1) and the constant coefficients $M_{kj}^{(1)}$, $M_{kj}^{(2)}$, M_{kj} depend on the vector of launching conditions a' and are obtained by numerical integration of the equations of unperturbed motion of an unperturbed missile.

Setting $k = 2$ in the approximate equalities

$$w_k(T) = w_k'(T) + \Delta w_k(T) \approx w_k'(T') + f_k(w'(T'))\Delta T + \Delta w_k(T) \quad (k = 1, 2, 3) \tag{3.2}$$

where f_k are the right-hand sides of the first three equations of unperturbed motion $(w_k')^{\bullet} = f_k(w')$, and noting that $w_2(T) = w_2'(T') = 0$ from the definition of T, T', we obtain $\Delta T = -\Delta w_2(T)/f_2(w'(T'))$. Substituting this expression for ΔT and expressions (3.1) for $\Delta w_k(T)$ into equalities (3.2) for $k = 1, 3$, we obtain a formula of the form (3.1) for the perturbations $\Delta w_k = w_k(T) - w_k'(T')$ of the coordinates of the impact point x, z.

In what follows, the set of all possible unperturbed launching conditions for missiles of this type is approximated by the finite set of vectors a_l' $(l = 1, \ldots, m)$. Then denoting by $N_{klj}^{(1)}$, $N_{klj}^{(2)}$, N_{klj} the coefficients of the formula for Δw_k evaluated at $a' = a_l'$, we find the perturbations of the coordinates of the impact point x and z given the launching conditions a_l'

$$\Delta w_{kl} = \sum_{j=1}^{11} (N_{klj}^{(1)}\epsilon_j^{(1)} + N_{klj}^{(2)}\epsilon_j^{(2)}) + \sum_{j=12}^{18} N_{klj}\epsilon_j \quad (k = 1, 3) \tag{3.3}$$

4. THE DIRECT DETERMINISTIC SCATTER PROBLEM (DDSP)

The set of perturbations (2.1) consists of the components of two-dimensional perturbations $\epsilon_j = (\epsilon_j^{(1)}, \epsilon_j^{(2)})$ $(j = 1, \ldots, 11)$ and one-dimensional perturbations ϵ_j $(j = 12, \ldots, 18)$. The tolerance d_j of the perturbation ϵ_j $(j = 1, \ldots, 18)$ is the maximum value of $|\epsilon_j|$ that occurs in the manufacturing and launching of a missile of the given type. The tolerance vector $d = (d_1, \ldots, d_{18})$ is considered as a point in the Euclidean space E^{18}. The quantities $|\epsilon_j| = [(\epsilon_j^{(1)})^2 + (\epsilon_j^{(2)})^2]^{1/2}$ $(j = 1, \ldots, 11)$ have an obvious physical meaning: $|\epsilon_1|$ is the magnitude of the wind velocity perturbations; $|\epsilon_2|$ and $|\epsilon_3|$ are the angles that the thrust vector and the first principal axis of inertia make with the longitudinal axis at time t_*, $|\epsilon_6|$ and $|\epsilon_7|$ are the corresponding angles at time τ, $|\epsilon_4|$ and $|\epsilon_5|$ are the distances from the point K and the centre of mass to the longitudinal axis at time t_*, $|\epsilon_8|$ and $|\epsilon_9|$ are the corresponding distances at time τ, $|\epsilon_{10}|$ is the magnitude of the tranverse angular velocity, and $|\epsilon_{11}|$ is the angle of attack at time t_0.

The DDSP involves determining D_k $(k = 1, 3)$—the maximum magnitudes of the perturbations of the point of impact coordinates x, z for given tolerances and launching conditions. According to (3.3) and the definition of d, the maximum magnitude of the perturbations of the point of impact coordinates x, z given the launching conditions a_l' is $D_{kl} = n_{kl}d$, where n_{kl} is a vector with the components

$$n_{klj} = \sqrt{(N_{klj}^{(1)})^2 + (N_{klj}^{(2)})^2} \quad (j = 1, \ldots, 11), \; n_{klj} = |N_{klj}| \quad (j = 12, \ldots, 18)$$

and $n_{kl}d$ is the scalar product of n_{kl} and d. Thus, D_k $(k = 1, 3)$ is the largest of D_{kl} $(l = 1, \ldots, m)$.

5 THE INVERSE DETERMINISTIC SCATTER PROBLEM (IDSP)

Let L be the set of index pairs $k = 1, 3$, $l = 1, \ldots, m$ such that $n_{kl} \neq 0$. The perturbation ϵ_j is unimportant when $n_{klj} = 0$ for all $(k, l) \in L$. The IDSP involves finding the tolerance vector such that for all $l = 1, \ldots, m$ the perturbations of the point of impact coordinates x, z do not exceed in magnitude the given values $D_k > 0$ $(k = 1, 3)$, i.e.

$$n_{kl}d \leqslant D_k, \; (k, l) \in L \tag{5.1}$$

To avoid solutions with negative or very small tolerances d_j, whose realization is impossible or impracticable, we introduce the conditions

$$d \geqq d_0, \; d_0 = (d_{01}, \ldots, d_{018}) \tag{5.2}$$

where $d_{0j} \geq 0$ are the minimum realizable d_j. Here and in what follows $d' \geq d''$ when $d_j' \geq d_j''$ ($j = 1, \ldots, 18$), and $d' \geqq d''$, when $d_j' \geq d_j''$ ($j = 1, \ldots, 18$), and $d_j' > d_j''$ for at least one j.

If the vectors d' and d'' satisfy conditions (5.1) and (5.2) while some components of d' are greater than and some are less than the corresponding components of d'', then in the absence of additional information there is no reason to prefer one vector to the other. If, however, $d' \geqq d''$, then d' is preferable, because in this case for some perturbations ϵ_j the tolerances are greater than with d'' and for other perturbations the tolerances are equal.

We thus obtain the following formulation of the IDSP: find the tolerance vector d that satisfies inequalities (5.1) and (5.2) and such that there is no other vector $d' \geqq d$ that satisfies these inequalities. The IDSP thus reduces to finding the pareto-optimal solutions [3] of the problem of maximizing the function $f(d) \equiv d$ on the polyhedron X—the intersection of the unbounded convex polyhedra M and K defined by inequalities (5.1) and (5.2). The polyhedron K is the positive coordinate orthant $d \geq 0$ shifted by d_0. It is thus a cone with its apex at d_0. The part of M included in the orthant $d \geq 0$ is bounded.

6. THE PROPERTIES OF IDSP SOLUTIONS AND THE METHOD OF SOLUTION

The n-face of the polyhedron M is the set $\Gamma_n(k_1, l_1, \ldots, k_n, l_n)$ of points $d \in M$ contained in the intersection of n hyperplanes $n_{kl}d = D_k$, $(k, l) \in \{(k_1, l_1), \ldots, (k_n, l_n)\}$ with pairwise distinct $(k_s, l_s) \in L$ ($s = 1, \ldots, n$) and not contained in the intersection of any $n+1$ such hyperplanes. The interior of M is treated as a 0-face Γ_0. The dimension of a non-empty n-face is $18-n$ or less than $18-n$ (when some of the defining hyperplanes are identical). Each point $d \in M$ is contained only in one n-face $\Gamma_n(k_1, l_1, \ldots, k_n, l_n)$, which we denote by $\Gamma(d)$; we denote by $L(d)$ the set $(k_1, l_1), \ldots, (k_n, l_n)$ of index pairs (k, l) for which $n_{kl}d = D_k$. With the exception of the vertices of M, all non-empty n-faces are open sets.

The n-face $\Gamma(d)$ is said to be parallel to the vector e if together with any segment $d' + \lambda e$ ($\lambda_1 < \lambda < \lambda_2$) parallel to e this n-face contains the point d'. The set of indices j of the unit basis vectors $e^{(j)}$ ($j = 1, \ldots, 18$) to which the n-face $\Gamma(d)$ is parallel will be denoted by $J(d)$; it consists of the indices j of common zero components of the vectors n_{kl}, $(k, l) \in L(d)$. Since $n_{kl} \geq 0$, $(k, l) \in L$, the set $E(d)$ of the vectors $e \geq 0$ to which $\Gamma(d)$ is parallel consists of vectors with the components $e_j = 0$ ($j \notin J(d)$), $e_j \geq 0$ ($j \in J(d)$) related by the constraint $e_1 + \ldots + e_{18} = 1$. The last constraint is imposed to ensure non-collinearity of any distinct vectors from $E(d)$. Since the 0-face Γ_0 is parallel to all $e^{(j)}$ ($j + 1, \ldots, 18$), E_0 is the set of vectors e with the components $e_j \geq 0$, $e_1 + \ldots + e_{18} = 1$.

Let $d \in X$. The set of points $d' \geqq d$ (points that are "better" than d) is a cone with the punctured apex d, which is included in K and consists of the rays $d + \lambda e$ ($\lambda > 0$) pointing in the directions $e \geqq 0$. If $E(d)$ is empty, then d is a solution of IDSP, because all such rays lie outside M. Indeed, in this case for any $e \geqq 0$ a vector n_{kl}, $(k, l) \in L(d)$, such that $n_{kl}e > 0$ and thus $n_{kl}(d + \lambda e) = D_k + \lambda n_{kl}e > D_k$ exists. Assume that $E(d)$ is non-empty. If the vector $e \geqq 0$ is not contained in $E(d)$, then the ray $d + \lambda e$ ($\lambda > 0$) again is not contained in M. If $e \in E(d)$, then the ray $d + \lambda e$ ($\lambda > 0$) intersects the boundary of $\Gamma(d)$ for $\lambda > 0$ given by

$$\lambda(d, e) = \min_{\substack{(k,l) \in L \\ (k,l) \notin L(d)}} \lambda_{kl}(d, e); \quad \lambda_{kl}(d, e) = \begin{cases} (D_k - n_{kl}d)/n_{kl}e & (n_{kl}e > 0) \\ \infty & (n_{kl}e = 0) \end{cases}$$

For $\lambda > \lambda(d, e)$ the points of this ray are outside M, and for $0 < \lambda < \lambda(d, e)$ they are contained in $\Gamma(d)$, but then they are all "worse" than the point $d + \lambda(d, e)e$. This leads to two assertions.

1. The set P of solutions of the IDSP is non-empty only when $d_0 \in M$ (i.e. $n_{kl}d_0 \leq D_k$, $(k, l) \in L$) and it is the intersection of the cone K with all n-faces that are not parallel to any $e^{(j)}$ ($j = 1, \ldots, 18$).

2. Any solution of the IDSP can be computed as an element of the sequence $d_{(0)}, d_{(1)}, \ldots$ defined by the equalities

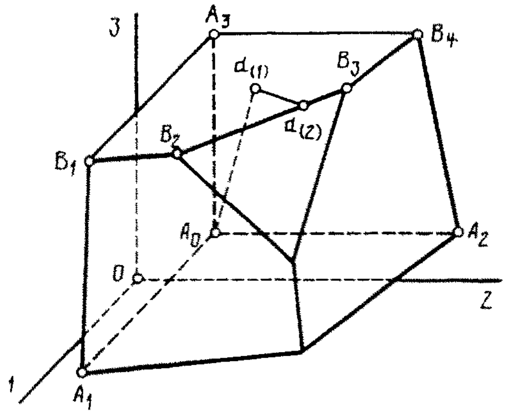

FIG. 1.

$$d_{(0)} = d_0 ; d_{(s+1)} = d_{(s)} + \lambda(d_{(s)}, \quad e_{(s)}) e_{(s)}, \quad e_{(s)} \in E(d_{(s)}) \tag{6.1}$$

for which $E(d_{(s)})$ is empty. The index s^* of this element is not greater than the dimensions of $\Gamma(d_0)$.

Setting $\lambda(d_0, e^{(i)}) = 0$ for $e^{(i)} \notin E(d_0)$, we consider the point of intersection $d^{(0i)} = d_0 + \lambda(d_0, e^{(i)}) e^{(i)}$ of the boundary $\Gamma(d_0)$ with the ray $d_0 + \lambda e^{(i)}$ ($\lambda \geqslant 0$). Its ith coordinate is $d_i^0 = d_{0i} + \lambda(d_0, e^{(i)})$ and the remaining coordinates are d_{0j} ($j \neq i$). Assume that a vector $d \in X$ exists for which $d_i > d_i^0$, and represent it as the sum $d = d' + d''$, where $d' = d_0 + \lambda' e^{(i)}$, and d'' is orthogonal to the ray $d_0 + \lambda e^{(i)}$ ($\lambda \geqslant 0$). Since $d_i' = d_i > d_i^0$, we have $\lambda' > \lambda(d_0, e^{(i)})$, and consequently $d' \notin M$, and since $d'' \geqslant 0$, also $d' + d'' \notin M$. We thus have the following assertion.

3. The tolerance d_i ($i = 1, \ldots, 18$) reaches its maximum value when the other tolerances d_{0j} ($j \neq i$) are minimal and this maximum value is $d_i^0 = d_{0i} + \lambda\,(d_0, e^{(i)})$.

Suppose that the boundary of M in the three-dimensional case is formed by four planes, one of which is parallel to $e^{(1)}$ and $e^{(2)}$, and the point $d_0 = A_0$ is contained inside M, so that X is the heptagon shown in Fig. 1. Then the set of solutions P is the surface formed by the three polygons shown by the thick lines. The maximum tolerance d_i^0 ($i = 1, 2, 3$) is equal to the ith coordinate of the point $d^{(0i)} = A_i$. Any solution $d \in P$ is at worst the second term $d_{(2)}$ of the sequence (6.1). If $d_0 = A_3$, then P is the polygonal line $B_1, \ldots, B_4$.

The components $e_{(s)j} \geqslant 0$, $j \in J(d_{(s)})$ of the vector $e_{(s)} \in E(d_{(s)})$ add up to 1 and therefore $0 \leqslant e_{(s)j} \leqslant 1$. The closer $e_{(s)j}$ in (6.1) is to 1, the greater is the increment of the tolerance d_j on passing from $d_{(s)}$ to $d_{(s+1)}$ and the smaller are the increments of the other tolerances. We therefore first find the maximum tolerances d_j^0 ($j = 1, \ldots, 18$) and choose the largest components of $e_{(0)}$ as those that correspond to hard to realize d_{j0}. Then $d_{(1)}$ is computed. The person solving the IDSP estimates the tolerance vector $d_{(1)}$ based on his view of the difficulty of its realization and other unformalized criteria. If the vector $d_{(1)}$ is "satisfactory", it is accepted as a solution (when $J(d_{(1)})$ is empty) or is "improved" by formulae (6.1) (when $J(d_{(1)})$ is non-empty). Otherwise, the vector $e_{(0)}$ is updated.

If we need a solution of the IDSP in which some of the tolerances take specified values, the remaining tolerances can be computed by the proposed method setting d_{0j} equal to the prescribed tolerances d_j and taking $e_{(s)j} = 0$ for the corresponding indices j.

7. THE DIRECT PROBABILISTIC SCATTER PROBLEM (DPSP)

Assume that each perturbation in (2.1) is a random variable independent of the other perturbations and has a normal distribution with mean 0. Each pair of perturbations $\epsilon_j^{(1)}$, $\epsilon_j^{(2)}$ ($j = 1, \ldots, 11$) is bivariate normal. Then the distribution of the perturbations (2.1) is completely characterized by the vector of standard deviations $\sigma = (\sigma_1, \ldots, \sigma_{18})$ or the vector of variances $d = (d_1, \ldots, d_{18})$, where $d_j = \sigma_j^2$ for $j = 1, \ldots, 11$ is the variance of $\epsilon_j^{(1)}$ and $\epsilon_j^{(2)}$ and for $j = 12, \ldots,$

18 it is the variance of ϵ_j. From (3.3) we thus have that for each $l = 1, \ldots, m$ any linear combination of Δw_{1l}, Δw_{3l} with constant coefficients that are not simultaneously zero has a univariate normal distribution. This means that the perturbations Δw_{1l}, Δw_{3l} of the point of impact coordinates x, z for a fixed a_l' follow a bivariate normal distribution with probability density

$$p(\Delta w_{1l}, \Delta w_{3l}) = (2\pi \Sigma_{1l}\Sigma_{3l}\sqrt{1-\rho_l^2})^{-1} \exp [-(\Delta w_{1l}^2/\Sigma_{1l}^2 - $$
$$- 2\rho_l \Delta w_{1l} \Delta w_{3l}/\Sigma_{1l}\Sigma_{3l} + \Delta w_{3l}^2)/\Sigma_{3l}^2/2(1-\rho_l^2)]$$

where $\Sigma_{kl} = \sqrt{D_{kl}}$ are the standard deviations of Δw_{kl} $(k = 1, 3)$, $\rho_l = C_l/\Sigma_{1l}\Sigma_{3l}$ is the correlation coefficient between them. The variances D_{kl} of the variables and their covariance C_l are obtained by (3.3) from the formulae

$$D_{kl} = n_{kl}d, \quad C_l = n_l d \tag{7.1}$$

where n_{kl}, n_l are vectors with the components

$$n_{klj} = \begin{cases} (N_{klj}^{(1)})^2 + (N_{klj}^{(2)})^2 \\ N_{klj}^2 \end{cases} \qquad n_{lj} = \begin{cases} N_{1lj}^{(1)}N_{3lj}^{(1)} + N_{1lj}^{(2)}N_{3lj}^{(2)}, & j = 1, \ldots, 11 \\ N_{1lj}N_{3lj}, & j = 12, \ldots, 18 \end{cases}$$

Instead of standard deviations, we may also consider the likely deviations $E_{kl} \approx 0.6745\Sigma_{kl}$ of the perturbations Δw_{kl} $(k = 1, 3)$.

If at the point of impact we change from the coordinates Δw_{3l}, Δw_{1l} to a right-hand system of coordinates with the axes Δw_{3l}^0, Δw_{1l}^0 directed, respectively, along the semiminor and semimajor axes of the equal probability ellipse, then the correlation coefficient between Δw_{1l}^0, Δw_{3l}^0 is zero and their variances are given by

$$D_{kl}^0 = \frac{1}{2}[D_{1l} + D_{3l} - (-1)^{(k+1)/2}\sqrt{(D_{1l} - D_{3l})^2 + 4C_l^2}] \quad (k = 1, 3)$$

The angle φ_l through which the axes Δw_{3l}, Δw_{1l} are rotated to coincide with Δw_{3l}^0, Δw_{1l}^0 is equal, apart from πs $(s = 0, \pm 1, \pm 2, \ldots)$, to φ_{1l} for $D_{1l} > D_{3l}$, φ_{2l}, for $D_{1l} < D_{3l}$, φ_{3l} for $D_{1l} = D_{3l}$ where $\varphi_{1l} = 0.5 \operatorname{arctg} 2C_l/(D_{3l} - D_{1l})$, $\varphi_{2l} = \varphi_{1l} + \pi/2$, $\varphi_{3l} = -0.25\pi \operatorname{sgn} C_l$ $(C_l \neq 0)$.

The missile scatter for a fixed a_l' is thus entirely characterized by one of the pairs of variables D_{kl}, D_{kl}^0, Σ_{kl}, Σ_{kl}^0, E_{kl}, E_{kl}^0 $(k = 1, 3)$ with one of the variables C_l, ρ_l, φ_l. To obtain a sufficiently complete picture of the scatter of a particular missile, we need to determine the maxima of the corresponding probabilistic characteristics over all $l = 1, \ldots, m$.

8. THE INVERSE PROBABILISTIC SCATTER PROBLEM (IPSP)

As the main missile scatter characteristics the maximum (over all launching conditions) standard range and lateral deviations Σ_1 and Σ_2 or the maximum likely range and lateral deviations E_1 = LR and E_3 = LL of the point of impact coordinates are often used. We will thus consider the IPSP as the problem of finding the variance vector d of the perturbations (2.1) such that for all $l = 1, \ldots, m$ the standard or likely range and lateral deviations do not exceed Σ_k or E_k $(k = 1, 3)$. Then, by (7.1), we have $n_{kl}d \leq D_k$, $(k, l) \in L$ where $D_k = \Sigma_k^2 \approx (E_k/0.6745)^2$ are given values, and L is the set of index pairs $k = 1, 3$, and $l = 1, \ldots, m$ for which $n_{kl} \neq 0$.

To avoid solutions with negative or very small variances d_j, we impose the condition $d \geq d_0$, where $d_0 \geq 0$ is the minimum variance vector. Thus, the vector d satisfies conditions that differ from (5.1) and (5.2) only by the definition of n_{kl}. Among such vectors, the solutions of the IPSP are those d for which no $d' \geq d$ exist. Then analogues of assertions 1–3 hold for IPSP.

Instead of d_j or σ_j we can use the quantiles $|\epsilon_j|_{0.997}$, which may be called the tolerances Δ_j at the 0.3% rejection level. Seeing that two-dimensional perturbations of $|\epsilon_j|^2/\sigma_j^2$ are distributed as $\chi^2(2)$, we have $\Delta_j \approx 3.6\sigma_j$ $(j = 1, \ldots, 11)$, $\Delta_j \approx 3\sigma_j (j = 12, \ldots, 18)$.

TABLE 1

j	PC-1 e_j	PC-1 Δ_j	PC-1 Δ_j^0	PC-2 e_j	PC-2 Δ_j	PC-2 Δ_j^0	Δ_j, Units of measurement
1	0.0153	0.52	3.6	0.0023	0.59	10	m/s
2	0.2703	0.074	0.12	0.2465	0.037	0.063	degrees
4.5	0.134	0.35	0.83	0.15	0.2	0.43	cm
6	0.0015	0.087	1.9	0.0011	0.043	1.1	degrees
8.9	0.0011	0.51	13	0.001	0.28	7.6	cm
10	0.038	0.51	2.2	0.055	0.34	1.2	degrees/s
11	0.13	0.5	1.2	0.098	0.35	0.95	degrees
12	0.0035	0.062	0.89	0.0034	0.055	0.8	degrees
13	0.0042	0.056	0.74	0.0047	0.058	0.71	degrees
14	0.19	0.19	0.37	0.19	0.16	0.3	kg/s
15	0.05	5.2	20	0.059	5.0	17	m/s
16	0.01	0.58	5.0	0.014	0.58	4.1	kg
17	0.005	0.57	6.9	0.007	0.56	5.6	kg
18	0.012	0.0038	0.03	0.017	0.0042	0.027	—

8. EXAMPLE

Missile M2 with a range of 100 km differs from missile M1 only in the design of the tail, that approximately halves the restoring moment coefficient and leads to some changes in the other characteristics. Table 1 gives the values of $e_j = e_{(0)j}$ chosen after a number of trials and the corresponding values of $\Delta_j = \Delta_{(1)j}$ calculated in the first step of the process (6.1) with $d_0 = 0$, LR = LL = 500 m, and also Δ_j^0. The perturbations ϵ_j ($j = 3, 7$) proved to be unimportant, and Δ_j ($j = 2, 4, 5, 6, 8, 9, 10, 11$) for M2 are almost half those for M1. If we take for M2 the same tolerances Δ_j as for M1, then as we see from the solution of the IPSP its LR and LL are a factor of 1.8 greater than for M1. If with the tolerances of M1 we increase the rotational velocity of M2 by a factor of 2.1, then LR = LL = 500 m.

REFERENCES

1. GANTMAKHER F. R. and LEVIN L. M., *Theory of the Flight of Unguided Missiles*. Fizmatgiz, Moscow, 1959.
2. DMITRIYEVSKII A. A., *Exterior Ballistics*. Mashinostroyeniye, Moscow, 1972.
3. PODINOVSKII V. V. and NOGIN V. D., *Pareto-optimal Solutions of Multicriterion Problems*. Nauka, Moscow, 1982.

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